EQUIVALENCE OF DOMAINS ARISING FROM DUALITY OF ORBITS ON FLAG MANIFOLDS II

TOSHIHIKO MATSUKI

Abstract. In [GM1], we defined a $G_{\mathbb{R}}$-$K_{\mathbb{C}}$ invariant subset $C(S)$ of $G_{\mathbb{C}}$ for each $K_{\mathbb{C}}$-orbit S on every flag manifold $G_{\mathbb{C}}/P$ and conjectured that the connected component $C(S)_0$ of the identity will be equal to the Akhiezer-Gindikin domain D if S is of nonholomorphic type. This conjecture was proved for closed S in [WZ1, WZ2, FH, M6] and for open S in [M6]. In this paper, we prove the conjecture for all the other orbits when $G_{\mathbb{R}}$ is of non-Hermitian type.

1. Introduction

Let $G_{\mathbb{C}}$ be a connected complex semisimple Lie group and $G_{\mathbb{R}}$ a connected real form of $G_{\mathbb{C}}$. Let $K_{\mathbb{C}}$ be the complexification in $G_{\mathbb{C}}$ of a maximal compact subgroup K of $G_{\mathbb{R}}$. Let $X = G_{\mathbb{C}}/P$ be a flag manifold of $G_{\mathbb{C}}$ where P is an arbitrary parabolic subgroup of $G_{\mathbb{C}}$. Then there exists a natural one-to-one correspondence between the set of $K_{\mathbb{C}}$-orbits S and the set of $G_{\mathbb{R}}$-orbits S' on X given by the condition:

$$S \leftrightarrow S' \iff S \cap S' \text{ is non-empty and compact} \tag{1.1}$$

([M4]). For each $K_{\mathbb{C}}$-orbit S we defined in [GM1] a subset $C(S)$ of $G_{\mathbb{C}}$ by

$$C(S) = \{x \in G_{\mathbb{C}} \mid xS \cap S' \text{ is non-empty and compact}\}$$

where S' is the $G_{\mathbb{R}}$-orbit on X given by (1.1).

Akhiezer and Gindikin defined a domain $D/K_{\mathbb{C}}$ in $G_{\mathbb{C}}/K_{\mathbb{C}}$ as follows ([AG]). Let $\mathfrak{g}_{\mathbb{R}} = \mathfrak{k} \oplus \mathfrak{m}$ denote the Cartan decomposition of $\mathfrak{g}_{\mathbb{R}} = \mathrm{Lie}(G_{\mathbb{R}})$ with respect to K. Let $\mathfrak{t}$ be a maximal abelian subspace in $i\mathfrak{m}$. Put

$$\mathfrak{t}^+ = \{Y \in \mathfrak{t} \mid |\alpha(Y)| < \frac{\pi}{2} \text{ for all } \alpha \in \Sigma\}$$

where Σ is the restricted root system of $\mathfrak{g}_{\mathbb{C}}$ with respect to $\mathfrak{t}$. Then D is defined by

$$D = G_{\mathbb{R}}(\exp \mathfrak{t}^+)K_{\mathbb{C}}.$$

We conjectured the following in [GM1].

Conjecture 1.1. (Conjecture 1.6 in [GM1]) *Suppose that $X = G_{\mathbb{C}}/P$ is not $K_{\mathbb{C}}$-homogeneous. Then we will have $C(S)_0 = D$ for all $K_{\mathbb{C}}$-orbits S of nonholomorphic type on X. Here $C(S)_0$ is the connected component of $C(S)$ containing the identity. (See* [GM1, M6] *for the definition of the $K_{\mathbb{C}}$-orbits of nonholomorphic type. When $G_{\mathbb{R}}$ is of non-Hermitian type, all the $K_{\mathbb{C}}$-orbits are defined to be of nonholomorphic type.)*

Let $S_{\rm op}$ denote the unique open $K_{\mathbb C}$-B double coset in $G_{\mathbb C}$ where B is a Borel subgroup of $G_{\mathbb C}$ contained in P. It is shown in [H] and [M5] that $D\subset C(S_{\rm op})_0$. (The opposite inclusion $D\supset C(S_{\rm op})_0$ is proved in [B].) On the other hand the inclusion $C(S_{\rm op})_0\subset C(S)_0$ for every $K_{\mathbb C}$-orbit S on $X=G_{\mathbb C}/P$ is shown in [GM1] Proposition 8.1 and Proposition 8.3. So we have the inclusion

$$D\subset C(S)_0. \tag{1.2}$$

We have only to prove the opposite inclusion.

For a simple root α with respect to B we can define a parabolic subgroup P_α by

$$P_\alpha=B\sqcup Bw_\alpha B$$

where w_α is the reflection for the root α. Let S_0 be a closed $K_{\mathbb C}$-B double coset in $G_{\mathbb C}$. Let $S_1,\ldots,S_\ell$ ($\ell=$ codim $_{\mathbb C}S_0$) be a sequence of $K_{\mathbb C}$-B double cosets in $G_{\mathbb C}$ and $\alpha_1,\ldots\alpha_\ell$ a sequence of simple roots such that

$$S_k^{cl}=S_0P_{\alpha_1}\cdots P_{\alpha_k}$$

and that

$$\dim_{\mathbb C}S_k=\dim_{\mathbb C}S_0+k$$

for $k=1,\ldots,\ell$ (c.f. [GM2], [M3], [Sp]). Especially $S_\ell=S_{\rm op}$.

In this paper we first prove the following theorem.

Theorem 1.2. *Let x be an element of $G_{\mathbb C}$. If $I_0=xS_0\cap S'_{\rm op}P_{\alpha_\ell}\cdots P_{\alpha_1}$ is connected, then*

$$I_k=xS_k^{cl}\cap S'_{\rm op}P_{\alpha_\ell}\cdots P_{\alpha_{k+1}}$$

is connected for $k=1,\ldots,\ell$. ($S'_{\rm op}$ is the unique closed $G_{\mathbb R}$-B double coset in $G_{\mathbb C}$ which corresponds to $S_{\rm op}$ by (1.1).)

Remark 1.3. The sets I_k ($k=0,\ldots,\ell$) are always nonempty because $xS_0P_{\alpha_1}\cdots P_{\alpha_\ell}=xS_{\rm op}^{cl}=G_{\mathbb C}\supset S'_{\rm op}$.

Let S be a $K_{\mathbb C}$-P double coset in $G_{\mathbb C}$. Then we can write

$$S^{cl}=S_k^{cl}=S_0P_{\alpha_1}\cdots P_{\alpha_k}$$

with some closed $K_{\mathbb C}$-B double coset S_0 and a sequence $\alpha_1,\ldots,\alpha_k$ of simple roots ([M3], [Sp]). Secondly we prove the following.

Theorem 1.4. (i) *If $x\in D^{cl}$, then I_k is connected.*

(ii) *If $x\in D^{cl}\cap C(S)$, then $I_k=xS\cap S'_k$.*

As a corollary we solve Conjecture 1.1 for non-Hermitian cases:

Corollary 1.5. *Let $G_{\mathbb R}$ be simple and of non-Hermitian type. Then $C(S)_0=D$ for all the $K_{\mathbb C}$-orbits $S\neq X$ on $X=G_{\mathbb C}/P$.*

Proof. When S is open in $G_{\mathbb C}$, the equality is proved in [M6]. So we may assume that S is not open. Let x be an element of $D^{cl}\cap C(S)$. Then we have only to show that $x\in D$. Since $S_kP_{\alpha_{k+1}}\cdots P_{\alpha_{\ell-1}}\cap S_{\rm op}=\phi$, we have $S'_kP_{\alpha_{k+1}}\cdots P_{\alpha_{\ell-1}}\cap S'_{\rm op}=\phi$ by the duality ([M2]) and therefore

$$S'_{\rm op}P_{\alpha_{\ell-1}}\cdots P_{\alpha_{k+1}}\cap S'_k=\phi.$$

By Theorem 1.4 (ii) we have

$$\begin{aligned} xS^{cl} \cap S'_{\mathrm{op}} P_{\alpha_{\ell-1}} \cdots P_{\alpha_{k+1}} &= xS^{cl} \cap S'_{\mathrm{op}} P_{\alpha_\ell} \cdots P_{\alpha_{k+1}} \cap S'_{\mathrm{op}} P_{\alpha_{\ell-1}} \cdots P_{\alpha_{k+1}} \\ &= xS \cap S'_k \cap S'_{\mathrm{op}} P_{\alpha_{\ell-1}} \cdots P_{\alpha_{k+1}} = \phi. \end{aligned}$$

Hence

$$xS^{cl}_{\ell-1} \cap S'_{\mathrm{op}} = xS^{cl} P_{\alpha_{k+1}} \cdots P_{\alpha_{\ell-1}} \cap S'_{\mathrm{op}} = \phi.$$

For the orbit $S_{\ell-1}$ we defined the following domain Ω in [GM2].

$$\Omega = \{x \in G_{\mathbb{C}} \mid xS^{cl}_{\ell-1} \cap S'_{\mathrm{op}} = \phi\}_0.$$

It is shown in [FH] Theorem 5.2.6 and [M6] Corollary 1.8 that

$$\Omega = D$$

when $G_{\mathbb{R}}$ is of non-Hermitian type. Hence $x \in D$. $\square$

Remark 1.6. Recently [M7] proved Conjecture 1.1 for all non-closed $K_{\mathbb{C}}$-orbits in Hermitian cases using Theorem 1.4. Thus the conjecture is now completely solved affirmatively.

2. $G_{\mathbb{R}}$-orbits on the full flag manifold

The full flag manifold $\mathcal{F}$ of $G_{\mathbb{C}}$ is the set of the Borel subgroups of $G_{\mathbb{C}}$. If we take a Borel subgroup B_0 of $G_{\mathbb{C}}$, then the factor space $G_{\mathbb{C}}/B_0$ is identified with $\mathcal{F}$ by the map

$$G_{\mathbb{C}}/B_0 \ni gB_0 \mapsto gB_0g^{-1} \in \mathcal{F}.$$

It is known that every $G_{\mathbb{R}}$-orbit ($G_{\mathbb{R}}$-conjugacy class) on $\mathcal{F}$ contains a Borel subgroup of the form

$$B = B(\mathfrak{j}, \Sigma^+) = \exp\left(\sum_{\alpha \in \Sigma^+ \sqcup \{0\}} \mathfrak{g}_{\mathbb{C}}(\mathfrak{j}, \alpha)\right)$$

where $\mathfrak{j}$ is a θ-stable Cartan subalgebra of $\mathfrak{g}_{\mathbb{R}}$, Σ^+ is a positive system of the root system Σ of the pair $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{j}_{\mathbb{C}})$ and $\mathfrak{g}_{\mathbb{C}}(\mathfrak{j}, \alpha) = \{X \in \mathfrak{g}_{\mathbb{C}} \mid [Y, X] = \alpha(Y)X \text{ for all } Y \in \mathfrak{j}\}$ ([A], [M1], [R]).

Roots in Σ are usually classified as follows.

(i) If $\theta(\alpha) = \alpha$ and $\mathfrak{g}_{\mathbb{C}}(\mathfrak{j}, \alpha) \subset \mathfrak{k}_{\mathbb{C}}$, then α is called a "compact root".
(ii) If $\theta(\alpha) = \alpha$ and $\mathfrak{g}_{\mathbb{C}}(\mathfrak{j}, \alpha) \subset \mathfrak{m}_{\mathbb{C}}$, then α is called a "noncompact root".
(iii) If $\theta(\alpha) = -\alpha$, then α is called a "real root".
(iv) If $\theta(\alpha) \neq \pm\alpha$, then α is called a "complex root".

For a simple root α of Σ^+ define the parabolic subgroup P_α as in Section 1. By the same arguments as in [V] Lemma 5.1 and [M3] Lemma 3, we can prove the following decomposition of $P_\alpha/B \cong P^1(\mathbb{C})$ into the $P_\alpha \cap G_{\mathbb{R}}$-orbits.

Lemma 2.1. (i) *If α is compact, then $P_\alpha = (P_\alpha \cap G_{\mathbb{R}})B$.*

(ii) *If α is noncompact or real, then $P_\alpha/B \cong P^1(\mathbb{C}) = \mathbb{C} \sqcup \{\infty\}$ is decomposed into the three $(P_\alpha \cap G_{\mathbb{R}})_0$-orbits H_+, H_- and H_0 which are diffeomorphic to the upper half plane, the lower half plane and $P^1(\mathbb{R}) = \mathbb{R} \sqcup \{\infty\}$, respectively.* (*Sometimes H_+ and H_- are in the same $P_\alpha \cap G_{\mathbb{R}}$-orbit.*)

(iii) *If α is complex, then P_α/B is decomposed into the two $P_\alpha \cap G_{\mathbb{R}}$-orbits consisting of a point yB and the complement $(P_\alpha - yB)/B$.*

Remark 2.2. Concerning the $K_{\mathbb{C}}$-action on $G_{\mathbb{C}}/B$, it is shown in [V] Lemma 5.1 (c.f. [M3] Lemma 3, [GM1] Lemma 9.1) that:
(i) If α is compact, then $P_\alpha = (P_\alpha \cap K_{\mathbb{C}})B$.
(ii) If α is noncompact or real, then P_α/B is decomposed into three $(P_\alpha \cap K_{\mathbb{C}})_0$-orbits consisting of two points and the complement.
(iii) If α is complex, then P_α/B is decomposed into two $P_\alpha \cap K_{\mathbb{C}}$-orbits consisting of a point and the complement.

As a corollary of Lemma 2.1 we have:

Corollary 2.3. *Let g be an arbitrary element of $G_{\mathbb{C}}$. Then every $(gP_\alpha g^{-1} \cap G_{\mathbb{R}})_0$-invariant closed subset of gP_α/B is connected.*

Remark 2.4. On the contrary a $gP_\alpha g^{-1} \cap K_{\mathbb{C}}$-invariant closed subset of gP_α/B may not be connected in view of Remark 2.2 (ii).

3. Proof of the theorems

Proof of Theorem 1.2. We will prove the theorem by induction on k. Suppose that I_{k-1} is connected. Then

$$\begin{aligned} I_{k-1}P_{\alpha_k} &= (xS_{k-1}^{cl} \cap S'_{\mathrm{op}} P_{\alpha_\ell} \cdots P_{\alpha_k})P_{\alpha_k} \\ &= xS_k^{cl} \cap S'_{\mathrm{op}} P_{\alpha_\ell} \cdots P_{\alpha_k} \\ &= (xS_k^{cl} \cap S'_{\mathrm{op}} P_{\alpha_\ell} \cdots P_{\alpha_{k+1}})P_{\alpha_k} \\ &= I_k P_{\alpha_k} \end{aligned}$$

is connected. Suppose that $I_k = A_1 \sqcup A_2$ with some nonempty closed subsets A_1 and A_2 of I_k. Then we will get a contradiction. Since the Borel subgroup B is connected, A_1 and A_2 are right B-invariant. Since $A_1 P_{\alpha_k}$ and $A_2 P_{\alpha_k}$ are closed and

$$A_1 P_{\alpha_k} \cup A_2 P_{\alpha_k} = I_k P_{\alpha_k}$$

is connected, we have $A_1 P_{\alpha_k} \cap A_2 P_{\alpha_k} \neq \phi$. Take an element g of $A_1 P_{\alpha_k} \cap A_2 P_{\alpha_k}$. Then $gP_{\alpha_k} \cap I_k$ is decomposed as

$$gP_{\alpha_k} \cap I_k = (gP_{\alpha_k} \cap A_1) \sqcup (gP_{\alpha_k} \cap A_2)$$

with two nonempty closed subsets $gP_{\alpha_k} \cap A_1$ and $gP_{\alpha_k} \cap A_1$. But this contradicts Corollary 2.3 because $gP_{\alpha_k} \cap I_k = gP_{\alpha_k} \cap S'_{\mathrm{op}} P_{\alpha_\ell} \cdots P_{\alpha_{k+1}}$ is $gP_{\alpha_k} g^{-1} \cap G_{\mathbb{R}}$-invariant. □

Lemma 3.1. (i) *S_k is relatively closed in $S_{\mathrm{op}} P_{\alpha_\ell} \cdots P_{\alpha_{k+1}}$.*
(ii) *S'_k is relatively open in $S'_{\mathrm{op}} P_{\alpha_\ell} \cdots P_{\alpha_{k+1}}$.*

Proof. By the duality for the closure relation ([M3]) we have only to show (i). Let $\widetilde{S}$ be a $K_{\mathbb{C}}$-B double coset contained in the boundary of S_k. Then

$$\operatorname{codim}_{\mathbb{C}} \widetilde{S} > \operatorname{codim}_{\mathbb{C}} S_k = \ell - k.$$

Hence $\widetilde{S}$ cannot be contained in $S_{\mathrm{op}}P_{\alpha_\ell}\cdots P_{\alpha_{k+1}}$ by [V] Lemma 5.1 (c.f. [GM1] Lemma 9.1). $\square$

Proof of Theorem 1.4. (i) Since S_0/B is compact and S_0'/B is open, we see that

$$\begin{aligned} C(S_0) &= \{x \in G_{\mathbb{C}} \mid xS_0 \cap S_0' \text{ is nonempty and closed in } G_{\mathbb{C}}\} \\ &= \{x \in G_{\mathbb{C}} \mid xS_0 \subset S_0'\}. \end{aligned}$$

Hence $C(S_0)_0$ is the cycle space for S_0' defined in [WW]. Since $D \subset C(S_0)_0$ by (1.2), it follows that

$$x \in D \Longrightarrow xS_0 \subset S_0'.$$

Suppose that $x \in D^{cl}$. Then we have

$$xS_0 \subset {S_0'}^{cl} \subset S_{\mathrm{op}}'P_{\alpha_\ell}\cdots P_{\alpha_1}$$

and hence $I_0 = xS_0$ is connected. By Theorem 1.2 the intersection

$$I_k = xS^{cl} \cap S_{\mathrm{op}}'P_{\alpha_\ell}\cdots P_{\alpha_{k+1}}$$

is connected.

(ii) By Lemma 3.1 S_k' is relatively open in $S_{\mathrm{op}}'P_{\alpha_\ell}\cdots P_{\alpha_{k+1}}$. On the other hand S is also relatively open in S^{cl}. Hence

$$xS \cap S_k' \text{ is relatively open in } I_k = xS^{cl} \cap S_{\mathrm{op}}'P_{\alpha_\ell}\cdots P_{\alpha_{k+1}}. \tag{3.1}$$

Suppose that $x \in C(S)$. Then $xS \cap S'$ is nonempty and closed in $G_{\mathbb{C}}$ by definition. Since S_k' is relatively closed in S', it follows that

$$xS \cap S_k' \text{ is closed in } G_{\mathbb{C}}. \tag{3.2}$$

Since $xS \cap S' = (xS \cap S_k')P$, it also follows that

$$xS \cap S_k' \text{ is nonempty.} \tag{3.3}$$

Suppose moreover that $x \in D^{cl}$. Then I_k is connected by (i). Hence it follows from (3.1), (3.2) and (3.3) that

$$I_k = xS \cap S_k'. \qquad \square$$

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Department of Mathematics, Faculty of Science, Kyoto University, Kyoto 606-8502, Japan

E-mail address: matsuki@math.kyoto-u.ac.jp